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Development in understanding of Gauss-Krüger projection and its outcomes

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Abstract: The role of Gauss and Krüger is made clear in developing Gauss-Krüger projection. Gauss developed the projection and Krüger had brought Gauss's posthumous work into the open. From studying such historical issues, useful projection formula was found and this is now implemented for actual usage in surveying in Japan.

Keywords: Gauss, Gauss-Krüger, Map projection, Formula, History

1. Introduction

Gauss-Krüger projection is an ellipsoidal form of transverse Mercator projection and is widely used as the basis of UTM projection and many national grid systems including Japanese Plane Rectangular Coordinate Systems.

Notwithstanding the importance of this projection, there had been wrong views in Japan concerning the history of the development of this projection. They said that Gauss only gave basic ideas of the projection and Krüger developed its mathematical formulas. Another misunderstanding was Gauss conformal double projection (i.e. Gauss-Schreiber projection) was developed by Gauss and it was used in his Hanover survey (Japan Cartographers Association, 1998). But Gauss actually developed the mathematics of the Gauss-Krüger projection and used it in his survey for processing survey results.

The author demonstrated based on the original literature that Gauss developed the formulas already in 1820s and Krüger had brought Gauss's posthumous work into the open (Masaharu, 2000). The author later noticed this kind of misapprehension is not limited to Japanese literature but some English literature have also misleading statements on this point.

Another point the author would like to discuss in this paper is the meaning of Krüger's 1912 paper. It is often cited as the fundamental literature on this projection. The author had been seeking this paper since around 2000 but had not been able to find it long. In 2007 the author found that the library of GeoForschungsZentrum GFZ in Potsdam possess the paper and obtained the copy of the paper in PDF thanks to the kindness of the library. After reading through the paper it became clear that the objective of the paper was not to explain the projection comprehensively but to develop useful formulas for the calculation of wide area expanding up to nine degree longitudinal difference from the central meridian aiming at covering entire German territory in one coordinate system. In this sense it would not be appropriate to simply cite this paper as the basic literature on Gauss-Krüger projection.

Nevertheless Krüger's 1912 paper is a valuable and significant contribution to the study of this projection. The author found that the first formula written in this paper, which was originally developed by Gauss and fully developed by Krüger, has very suitable characteristics for use with a computer (Masaharu, 2008a). Therefore this Krüger's formula is recently paid attention to and used as a basis to develop formulas with higher accuracy (Karney, 2011; Kawase, 2011; Kawase, 2013).

2. Misunderstandings seen in the history of Gauss-Krüger projection and Gauss conformal double projection

2.1 Situation in Japan

As written before, there had been misunderstanding about the history of Gauss-Krüger projection in Japan and it had been authorized by descriptions in a technical term dictionary (Japan Cartographers Association, 1998). It is said that Gauss only gave basic ideas and Krüger developed mathematical formulas of this projection. Further it is said that Gauss conformal double projection was developed and used by Gauss for his Hanover survey.

The cause of this misunderstanding is not clear but there was a possibility that the order of the usage of these two projections might affect the idea. The Gauss conformal double projection (Gauss-Schreiber projection) had been used in Japan as the old Plane Rectangular Coordinate System for the processing of geodetic survey data since 19th century until 1952 when it was replaced by the new coordinate system based on Gauss-Krüger projection. This historical order is also the same in Germany.

The Gauss conformal double projection is based on the Gauss's paper (Gauss, 1844) that describes the method to map the ellipsoid surface onto sphere conformally with low length distortion. But application of this method to the map projection onto a plane for geodetic survey was developed by Oscar Schreiber in the latter half of the 19th

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century. Therefore Gauss is exterior to this projection system.

There were a few literatures in Japan that correctly describe the history of the Gauss-Krüger projection but there seemed no arguments between these contradicting views before the author raised the question. Masaharu (2000) demonstrated that Gauss developed the so-called Gauss-Krüger projection and used it for his Hanover survey mainly based on Gauss's works (Gauss, posthumous; Gauss, 1844) and made it clear Krüger was the editor of the Gauss posthumous works. Gauss (posthumous) includes the well-known formula of Gauss-Krüger projection with series expansion of the longitudinal difference from the central meridian. This is the decisive evidence that Gauss developed this projection. The introduction of Gauss (1844) also suggests that the method written in Gauss (1844) is different from the method used in his Hanover survey.

After this paper (Masaharu, 2000) is published, many meetings of the special interest group on cartography terms of Japan Cartographers Association were held to discuss whether descriptions of these projections in the dictionary should be changed. After long discussions these description were decided to be changed according to the author's opinion and errata are published.

2.2 Descriptions in English and German literatures

In order to solve the above-mentioned problem, the author looked for many documents that include descriptions about the developer of the Gauss-Krüger projection. Generally speaking, German books or papers describes clearly that Gauss developed it and used it for his survey. But descriptions in English books does not necessarily clear on this point.

Snyder (1993) is a very valuable contribution to the study of map projections and very useful reference in histo-ry of map projections. The author have learned a lot from this book. But even this book writes in the following way about the history of the Gauss-Krüger projection.

"Gauss laid the foundations and provided the equations used for the Gauss conformal projection of the nine-teenth and twentieth centuries (Lee 1976, 100-101). Louis Krüger (1912) derived refinements leading to the com-monly used Gauss-Krüger projection." (Snyder, 1993, p.160)

From this part, one may consider that Gauss conformal and Gauss-Krüger projection are different if the "re-finements" are interpreted considerable. Snyder (1993) is accurate and correct, the author consider, but this para-graph might lead to some misunderstanding where erroneous views are prevailing like in former Japan.

Bugayevskiy and Snyder (1995) writes as follows.

> 5.1.3 *Gauss-Krüger projection*
>
> In 1820-1830 Gauss developed and published a double conformal projection preserving scale along the central meridian, and it was used in practice for calculating the Hannover triangulation. The theory of this projection was also published by Oskar Schreiber in 1866. Detailed investigations of this projection were conducted by Louis Krüger and described by him in 1912 and 1919. He suggested a method of direct transformation of the ellipsoid onto the plane instead of the above double projection. Since then this pro-jection has been called the Gauss-Krüger projection.
>
> The Gauss-Krüger projection is determined by three conditions: it is conformal, it is symmetric about the central meridian, and it preserves linear scale along the central meridian.

(Bugayevskiy and Snyder, 1995, p.159-160)

This description has a few problems. Firstly Gauss-Krüger projection is not normally characterized as a double conformal projection. This sentence correctly states the projection preserves scale along the central meridian and was used for Hanover survey. Nevertheless Gauss method is again characterized as the double projection in contrast to the Krüger's direct transformation. This is not true because Krüger (1912, p.III-IV) writes that Gauss studied extensively double conformal projections but the projection actually applied for his survey was direct transformation. This means Krüger says that Gauss's projection for his survey was direct transformation. It should be emphasized that Gauss conformal projection used for his Hanover survey and Krüger's projection including several new formulas are in principle the same projection. The first sentence of this citation reminds the author about the misunderstanding in Japan that Gauss used a double conformal projection.

Secondly the word "published" in the first sentence is erroneous. Many other documents state that Gauss did not publish the projection method used for his survey.

Thirdly "Detailed investigations of this projection were conducted by Louis Krüger and described by him in 1912 and 1919" is not appropriate because 1919 paper of Krüger is meant for giving practical formulas for the 3 degree longitudinal width coordinate systems.

3. The meaning of Krüger's 1912 paper and recent developments based on its contents

Krüger's 1912 paper "Konforme Abbildung des Erdellipsoids in der Ebene (Conformal projection of the earth el-lipsoid into the plane)" is often cited as the fundamental literature on Gauss-Krüger projection. It is right in a sense because the paper has rather comprehensive explanations about the derivation of formulas. And this is the original intention to write a paper on this projection. But Krüger had a big challenging issue to derive useful formulas for the calculation of such a wide area expanding up to nine degree longitudinal difference from the central meridian. At that time it was planned to change plane coordinate system that can cover entire Germany and Austria without length distortion along the central meridian in one coordinate system. The old coordinate system based on Gauss conformal double projection (Gauss-Schreiber projection) had the problem that length distortion occurs along the central meridian when applied area extending long in the north-south direction. Krüger planned to develop such formulas that covers areas with eighteen degree longitudinal width, with enough high precision and high calculation efficiency. He succeeded to develop such formulas and write them as the second formula in his 1912 paper. This is the main contribution of Krüger's 1912 paper. In this sense it would not be appropriate to simply cite this paper as the basic literature on Gauss-Krüger projection. It had a special purpose.

The one coordinate system covering such a wide area would be appropriate for the purpose of processing geodetic survey results. But it is not suitable for applying to cartography because of the large length distortion at areas far from the central meridian. In 1917 a commission of Prussian Land Survey discussed unification of coordinates of Germany and Austria and decided to adopt new coordinate system that divides the area into three degree longitudinal width zones and apply Gauss-Krüger projection to each zone. It is intended to apply this coordinate system not only to geodetic surveying but also to cadaster maps. Krüger's 1919 paper was written for explaining the simplified formula based on the above mentioned second formula of 1912 paper. The formula of 1919 paper should be applied these three degree width zones (Eggert, 1920).

Due to this situation, the second formula developed in Krüger's 1912 paper had scarcely been used. The author could not find formula of Krüger's 1919 paper in textbooks although it was intended for use for divided longitudinal zone systems. Classical formula of this projection is series expansion in longitude difference. This was already developed by Gauss and have been used very widely. For making it clear, this formula is given below. This is the form that had been long used in Japan.

$$x = N\lambda\cos\phi[1+\frac{\lambda^2}{6}\cos^2\phi\{(1-t^2+u^2)+\frac{\lambda^2}{20}\cos^2\phi(5-18t^2+t^4+14u^2-58t^2u^2)\}]$$

$$y = B+\frac{1}{2}N\lambda^2\sin\phi\cos\phi[1+\frac{\lambda^2}{12}\cos^2\phi\{(5-t^2+9u^2+4u^4)+\frac{\lambda^2}{30}\cos^2\phi(61-58t^2+t^4+270u^2-330t^2u^2)\}]$$

where,

$$t = \tan\phi$$

$$u^2 = \frac{e^2\cos^2\phi}{1-e^2}$$

$$N = a/\sqrt{1-e^2\sin^2\phi}$$

$$B = a[\left(1-\frac{e^2}{4}-\frac{3e^4}{64}-\frac{5e^6}{256}-\frac{175e^8}{16384}\right)\phi-\left(\frac{3e^2}{8}+\frac{3e^4}{32}+\frac{45e^6}{1024}+\frac{105e^8}{4096}\right)\sin 2\phi$$
$$+\left(\frac{15e^4}{256}+\frac{45e^6}{1024}+\frac{525e^8}{16384}\right)\sin 4\phi-\left(\frac{35e^6}{3072}+\frac{175e^8}{12288}\right)\sin 6\phi+\frac{315e^8}{131072}\sin 8\phi]$$

a: equatorial radius, e: eccentricity

Krüger (1912) contained one more formula, the first formula. This was originally developed by Gauss and fully developed by Krüger. The author found it has very suitable characteristics for use with a computer and is applicable to a rather wide area with high accuracy (Masaharu, 2008a). The author also disserted and gave explanations on the derivation of this formula in Masaharu (2008b) and Masaharu (2011). The formula is shown below. (Formulas for calculation of coefficients are omitted.)

$$\tan\left(\frac{\pi}{4}+\frac{b}{2}\right) = \tan\left(\frac{\pi}{4}+\frac{B}{2}\right)\left(\frac{1-e\sin B}{1+e\sin B}\right)^{e/2}$$

$$\eta' = \frac{1}{2}\ln\frac{1+\sin L\cos b}{1-\sin L\cos b}$$

(alternatively $\eta' = \tanh^{-1}(\sin L\cos b)$)

$$\tan\xi' = \frac{\tan b}{\cos L}$$

$$x = A\{\xi'+\gamma_1\sin 2\xi'\cosh 2\eta'+\gamma_2\sin 4\xi'\cosh 4\eta'+\gamma_3\sin 6\xi'\cosh 6\eta'+\gamma_4\sin 8\xi'\cosh 8\eta'+\Lambda\}$$
$$y = A\{\eta'+\gamma_1\cos 2\xi'\sinh 2\eta'+\gamma_2\cos 4\xi'\sinh 4\eta'+\gamma_3\cos 6\xi'\sinh 6\eta'+\gamma_4\cos 8\xi'\sinh 8\eta'+\Lambda\}$$

where B: latitude, L: longitude

At first this formula of Gauss-Krüger projection seemed to be long missed and not be used. But later it was found that Bugayevskiy and Snyder (1995) writes about this equation. Therefore this equation might have been used in some countries.

We can nowadays find a few studies on Gauss-Krüger projection based on this formula (Karney, 2011; Kawase, 2011a; Kawase, 2011b; Kawase, 2013). Karney (2011) implemented two algorithms, one is exact algorithm with it-eration and the other is based on Krüger (1912). He obtained a few nanometer accuracy in both methods.

Kawase (2011a, 2011b and 2013) derived general terms for the series expansion of Krüger's formula. This Kawase's formula was practically adopted in the calculation formulas of the Rules and Specifications of Public Sur-vey in Japan replacing the former formulas (series expansion of longitude difference from the central meridian) in 2013. This formula with up to fifth (in part sixth) order terms gives precise results for 15 digits decimal that corresponds to double precision floating number. It is enough to calculate up to the fourth order for surveying purposes as Krüger gave but calculation precise up to the last digit is preferred nowadays because the inverse calculation can give the same result as the original numbers.

4. Conclusions

The study started as searching who developed Gauss-Krüger projection. It became clear in the course that the history of Gauss-Krüger projection has very abundant contents and some knowledge obtained from the study can give useful hints to contemporary issues. One example is Krüger's first formula in 1912 paper.

There are many variations of projection formulas of Gauss-Krüger projection. To collect and analyze these and to study historically the usage of them in surveying organizations are issues left for further study.

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